SOLUTION OF THE EQUATIONS OF A REGULAR ELECTROSTATIC BEAM IN THE PRESENCE OF EMISSION FROM AN ARBITRARY SURFACE

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An analytic solution of the equations of a regular electrostatic beam in the presence of emission from an arbitrary surface under total space charge conditions is given. It is assumed that the emitter is a coordinate surface x^1 = const in an orthogonal system x^i (i = 1, 2, 3), and the emission-current density J is a given function $J(x^2, x^3)$. The solution is represented in the form of series in x^1 with coefficients that are functions of x^2, x^3 and determined from recurrence relations. In expansion along the length of an arc of the curvilinear axis x^1, which is orthogonal to the emitter, the first correction of the Child-Langmuir 3/2 law is determined only by the total curvature (the sum of the principal curvatures) of the emitting surface. Solution of the problem in the formulation in question permits determination of the collector shape that ensures the given distribution of the emission-current density over the given surface.

A regular* monoenergetic nonrelativistic beam of charged particles with the same value and sign of specific charge η in the absence of an external magnetic field is described in the stationary case by a system of differential equations, which in tensor form in an arbitrary curvilinear coordinate system x^i (i = 1, 2, 3) has the form

$$g^{ik}v_iv_k = 2\varphi, \qquad e^{ikl}\frac{\partial v_i}{\partial x^k} = 0,$$
$$\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ik}\rho v_k\right) = 0, \qquad \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ik}\frac{\partial\varphi}{\partial x^k}\right) = \rho, \quad (1)$$

where v_i are the covariant velocity components; φ is the scalar potential; ρ the space charge density; g_{ik} the covariant metric tensor; and $g = |g_{ik}|$ its determinant. Equations (1) are written in the dimensionless variables r°, V°, φ°, ρ° (r, V are the absolute values of the radius vector and velocity vector)

$$r = ar^\circ, \qquad V = UV^\circ,$$
$$\varphi = -\frac{U^2}{\eta}\varphi^\circ, \qquad \rho = \frac{U^2}{4\pi\eta a^2}\rho^\circ, \qquad (2)$$

where the symbol for a dimensionless value is omitted; a, U are constants, which have the dimension of length and velocity, respectively.

The first equation in (1) represents the energy integral; the second reflects the fact that the velocity is a potential vector; and the third and fourth are the equation of current conservation and the Poisson equation for the scalar potential.

Henceforth we shall assume that the emitting surface coincides with one of the surfaces x^1 = const of the orthogonal coordinate system x^i (i = 1, 2, 3). Without loss of generality, the constant can be assumed to equal zero. As is known, most interesting from a practical point of view are regimes with emission limited by the space charge: on the emitter $x^1 = 0$

$$V = 0, \quad \varphi = 0, \quad \partial\varphi / \partial x^1 = 0, \quad \rho v_{x^1} = J(x^2, x^3), \quad (3)$$

where $J(x^2, x^3)$ is the emission-current density, and v_{xi} are the physical velocity components.

Let us seek the solution of problem (1), (3) in the form of series in x^1 with coefficients that are functions of x^2, x^3:

$$v_1 = (x^1)^{2/3}\sum_{k=0}^{\infty}U_k(x^1)^k, \qquad v_2 = (x^1)^{\varkappa}\sum_{k=0}^{\infty}V_k(x^1)^k,$$
$$v_3 = (x^1)^{\varepsilon}\sum_{k=0}^{\infty}W_k(x^1)^k, \qquad 2\varphi = (x^1)^{4/3}\sum_{k=0}^{\infty}\varphi_k(x^1)^k,$$
$$2\sqrt{g}\,\rho = (x^1)^{-2/3}\sum_{k=0}^{\infty}\rho_k(x^1)^k, \qquad (4)$$

decomposing in similar series the elements of the metric tensor g_{ik}, g^{ik}, $\sqrt{g}$ and the combinations $\sqrt{g}\,g^{ik}$:

$$g_{11} = \sum_{k=0}^{\infty}a_k(x^1)^k, \qquad g_{22} = \sum_{k=0}^{\infty}b_k(x^1)^k, \qquad g_{33} = \sum_{k=0}^{\infty}c_k(x^1)^k,$$
$$g^{11} = \sum_{k=0}^{\infty}A_k(x^1)^k, \qquad g_{22} = \sum_{k=0}^{\infty}B_k(x^1)^k, \qquad g^{33} = \sum_{k=0}^{\infty}C_k(x^1)^k,$$
$$\sqrt{g} = \sum_{k=0}^{\infty}G_k(x^1)^k, \qquad \sqrt{g}\,g^{11} = \sum_{k=0}^{\infty}\alpha_k(x^1)^k,$$
$$\sqrt{g}\,g^{22} = \sum_{k=0}^{\infty}\beta_k(x^1)^k, \qquad \sqrt{g}\,g^{33} = \sum_{k=0}^{\infty}\gamma_k(x^1)^k. \qquad (5)$$

The coefficients G_k, α_k, β_k, γ_k can be expressed in terms of A_k, B_k, C_k or in terms of a_k, b_k, c_k.

If we have expansions for the covariant velocity components v_i, then it is easy to move to the physical components v_{xi} (h is a fixing index),

$$v_{xh} = \sqrt{g^{hh}}\,v_h.$$

Examination of the conditions for regularity of flow indicates that $\varkappa = \varepsilon = 5/3$ and makes it possible to express the coefficients of the expansions v_2 and v_3 in terms of U_k:

$$\left(\frac{5}{3}+k\right)V_k = \frac{\partial U_k}{\partial x^2} = U_{k2}',$$
$$\left(\frac{5}{3}+k\right)W_k = \frac{\partial U_k}{\partial x^3} = U_{k3}' \qquad (k = 0, 1, \ldots). \qquad (6)$$

Thus, when (3) is satisfied, the particles leave the emitter at a right angle to it [2].

*According to [1], we shall call the flow regular if the generalized momentum of the particle is a potential vector.

If we substitute the expressions for v_i into the first equation of (1), we obtain

$$\varphi_s = \sum_{k=0} \Big[\Big(U_k^2 + 2\sum_{l=1}^{k} U_{l-1}U_{2k-l+1}\Big) A_{s-2k} + \\ +\Big(2\sum_{l=0}^{k} U_l U_{2k-l+1}\Big) A_{s-2k-1} + \Big(V_k^2 + 2\sum_{l=1}^{k} V_{l-1}V_{2k-l+1}\Big) B_{s-2k-2} + \\ +\Big(2\sum_{l=0}^{k} V_l V_{2k-l+1}\Big) B_{s-2k-3} + \Big(W_k^2 + 2\sum_{l=1}^{k} W_{l-1}W_{2k-l+1}\Big)\times \\ \times C_{s-2k-2} + \Big(2\sum_{l=0}^{k} W_l W_{2k-l+1}\Big) C_{s-2k-3}\Big] \qquad (s=0,\ 1,\ \ldots). \qquad (7)$$

Summation with respect to k is controlled by the subscripts of A, B, C: for a fixed s, all values of k that give nonnegative subscripts are allowable. The coefficients with negative subscripts are by definition equal to zero.

If we use the Poisson equation, we find

$$\rho_t = \Big(t + \frac{1}{3}\Big) \sum_{s=0}^{t} \Big(s + \frac{4}{3}\Big) \varphi_s \alpha_{t-s} + \\ + \sum_{s=0}^{t-2} \Big[(\varphi_{s2}{}'\beta_{t-s-2})_2{}' + (\varphi_{s3}{}'\gamma_{t-s-2})_3{}'\Big], \\ (t = 0,\ 1,\ \ldots)\ . \qquad (8)$$

Here it should be borne in mind that the sum with respect to s from a to b is equal to zero if b < a.

In order to obtain relations that determine the coefficients of expansions (4), it remains to use the last condition of (3), which relates U_0 and J, and the equation of current conservation. If we equate the coefficients of identical powers of x^1, we have

$$p\sum_{t=0}^{p} \rho_t \sum_{l=0}^{p-t} A_l U_{p-t-l} + \sum_{t=0}^{p-2} \Big[\Big(\rho_t \sum_{l=0}^{p-t-2} B_l V_{p-t-l-2}\Big)_2' + \\ + \Big(\rho_t \sum_{l=0}^{p-t-2} C_l W_{p-t-l-2}\Big)_3'\Big] = 0, \\ U_0 = ({}^9/_2 a_0{}^{5/2} J)^{1/3} \qquad (p = 1,\ 2,\ \ldots)\ . \qquad (9)$$

Since a simple relation (6)–(8) exists between the coefficients of expansions (4), it is sufficient to examine one of series (4), for example, for the potential. Using (7)–(9), we find

$$\varphi_0 = \Big(\frac{9J}{2A_0}\Big)^{2/3}, \qquad \frac{\varphi_1}{\varphi_0} = -\frac{3}{5}\frac{A_1}{A_0} - \frac{8}{15}\frac{G_1}{G_0}, \\ \frac{\varphi_2}{\varphi_0} = \frac{1}{36}\Big(-\frac{A_2}{A_0} + \frac{U_1^2}{U_0^2} + \frac{B_0V_0^2 + C_0W_0^2}{\varphi_0}\Big) - \\ - \frac{1}{18}\Big(4\frac{\alpha_1}{\alpha_0} + 7\frac{\varphi_1}{\varphi_0}\Big)\Big(\frac{U_1}{U_0} + \frac{A_1}{A_0}\Big) - \\ - \frac{7}{72}\Big(4\frac{\alpha_2}{\alpha_0} + 7\frac{\alpha_1}{\alpha_0}\frac{\varphi_1}{\varphi_0}\Big) - \frac{(\beta_0\varphi_{02}')_2' + (\gamma_0\varphi_{03}')_3'}{8\alpha_0\varphi_0} - \\ - \frac{(\alpha_0 B_0 V_0 \varphi_0)_2' + (\alpha_0 C_0 W_0 \varphi_0)_3'}{36\alpha_0 A_0 U_0 \varphi_0}\ . \qquad (10)$$

As is known, the surface x^1 = const is characterized at each point by two of its principal curvatures $\varkappa_1$ and $\varkappa_2$ or the total curvature $T = \varkappa_1 + \varkappa_2$ and the Gaussian curvature $K = \varkappa_1\varkappa_2$. According to the Gauss theorem [3], K belongs to the internal geometry of the surface, i.e., it is completely determined by the assignment of the metrics on it:

$$K = \frac{1}{g_{22}g_{33} - (g_{23})^2}\Big[\frac{\partial^2 g_{23}}{\partial x^2 \partial x^3} - \frac{1}{2}\frac{\partial^2 g_{22}}{(\partial x^3)^2} - \\ - \frac{1}{2}\frac{\partial^2 g_{33}}{(\partial x^2)^2} + \Gamma_{23}^i\Gamma_{23}^j g_{ij} - \Gamma_{22}^i\Gamma_{33}^j g_{ij}\Big],$$

where Γ^i_{pk} is a Christoffel symbol of the second kind. Taking the convolution with respect to i, j and considering that the examination is being carried out in an orthogonal coordinate system, we obtain

$$K = \frac{1}{4g_{22}g_{33}}\Big\{-2\Big[\frac{\partial^2 g_{22}}{(\partial x^3)^2} + \frac{\partial^2 g_{33}}{(\partial x^2)^2}\Big] + g^{22}\Big[\Big(\frac{\partial g_{22}}{\partial x^3}\Big)^2 + \frac{\partial g_{22}}{\partial x^2}\frac{\partial g_{33}}{\partial x^2}\Big] + \\ + g^{33}\Big[\Big(\frac{\partial g_{33}}{\partial x^2}\Big)^2 + \frac{\partial g_{22}}{\partial x^3}\frac{\partial g_{33}}{\partial x^3}\Big]\Big\}.$$

Now if we use the conditions of a Euclidean space, which are expressed by equating the Riemann-Christoffel tensor to zero (six Lamé identities), we arrive at the following expression for the Gaussian curvature of the surface x^1 = const in orthogonal Euclidean coordinates x^i:

$$K = \frac{1}{4g_{11}}\frac{\partial \ln g_{22}}{\partial x^1}\frac{\partial \ln g_{33}}{\partial x^1}\ . \qquad (11)$$

On the other hand, for the total curvature T we have [4]

$$T = -\frac{1}{2\sqrt{g_{11}}}\Big(\frac{\partial \ln g_{22}}{\partial x^1} + \frac{\partial \ln g_{33}}{\partial x^1}\Big). \qquad (12)$$

From (11), (12) it is apparent that the principal curvatures $\varkappa_1$ and $\varkappa_2$ are determined by the expressions

$$\varkappa_1 = -\frac{1}{2\sqrt{g_{11}}}\frac{\partial \ln g_{22}}{\partial x^1}, \quad \varkappa_2 = -\frac{1}{2\sqrt{g_{11}}}\frac{\partial \ln g_{33}}{\partial x^1}\ . \qquad (13)$$

Let us dwell in more detail on the first two terms of the expansion of the potential. Bearing in mind that

$$\frac{G_1}{G_0} = \frac{1}{2}\frac{a_1}{a_0} - a_0^{1/2}T\ ,$$

we have

$$2\varphi = \Big(\frac{9}{2}J\Big)^{2/3} s^{4/3}\Big[1 + \Big(\frac{1}{3}\frac{a_1}{a_0^{3/2}} + \frac{8}{15}T\Big)s + \ldots\Big], \qquad (14)$$

where $s = a_0^{1/2}x^1$. The meaning of s will be explained subsequently. Note only that whereas the curvilinear coordinate x^1 can have any dimension when we return to dimensional values, s will have the dimension of length.

The first term of expansion (14) represents the well-known law of 3/2 for a plane diode [5, 6] in local notation [$J = J(x^2, x^3)$]. The correction to it, which is expressed by the second term, is a function of the properties of the surface itself (through its total curvature) as well as of the sense of the parameter in which the expansion is made. In order to explain this, we note that in a cylindrical diode with an emitter R = 1 and J = const, as x^1 we can use, for example,

$$(1°)\quad x^1 = R - 1, \qquad (2°)\quad x^1 = \ln R, \qquad (3°)\quad x^1 = 1 - R^{-1}.$$

Papers [7, 8] were devoted to the construction of expansions with x^1 in the form of (2°), (3°). For (1°)–(3°) we have, respectively,

$$(1°)\quad g_{11} = 1, \qquad (2°)\quad g_{11} = \exp(2x^1), \qquad (3°)\quad g_{11} = (1 - x^1)^{-4}.$$

Formula (14) takes on a universal form if the expansion is carried out along the length of the arc S of the curvilinear axis x^1:

$$S = \int \sqrt{g_{11}}\, dx^1. \tag{15}$$

If we expand the integrand in a series in x^1 and integrate, we obtain

$$S = a_0^{1/2} x^1 + \frac{1}{4}\frac{a_1}{a_0^{1/2}}(x^1)^2 + \frac{1}{6}\left(\frac{a_2}{a_0^{1/2}} - \frac{1}{4}\frac{a_1^2}{a_0^{3/2}}\right)(x^1)^3 + \ldots =$$
$$= s + \frac{1}{4}\frac{a_1}{a_0^{1/2}} s^2 + \frac{1}{6}\left(\frac{a_2}{a_0^2} - \frac{1}{4}\frac{a_1^2}{a_0^3}\right)s^3 + \ldots.$$

Thus, s is the principal term of the expansion of the length of the arc S in x^1. If we express s in terms of S

$$s = S - \frac{1}{4}\frac{a_1}{a_0^{3/2}} S^2 + \frac{1}{6}\left(\frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2}\right)S^3 + \ldots \tag{16}$$

and substitute (16) into (14), we have

$$2\varphi = (^9/_2 J)^{2/3} S^{4/3}\left(1 + {}^8/_{15} TS + \ldots\right). \tag{17}$$

When $S \to 0$, it is sufficient to limit the expansion to the first term, which represents the Child-Langmuir solution in the plane case (when T = 0 and J = = const). For cylindrical and spherical diodes, general expression (17) gives, respectively,

$$\frac{\varphi}{\varphi_0} = S^{4/3}\left(1 - \frac{8}{15} S + \ldots\right),$$
$$S = R - R_0, \quad R = (x^2 + y^2)^{1/2},$$
$$\frac{\varphi}{\varphi_0} = S^{4/3}\left(1 - \frac{16}{15} S + \ldots\right),$$
$$S = r - r_0, \quad r = (x^2 + y^2 + z^2)^{1/2}.$$

The next terms of the expansion can be treated similarly. Thus, on the basis of the definition of G_2, we obtain

$$\frac{\alpha_2}{\alpha_0} = \frac{G_2}{G_0} + \frac{G_1}{G_0}\frac{A_1}{A_0} + \frac{A_2}{A_0} = \frac{1}{4}\frac{a_1}{a_0^{1/2}} T +$$
$$+ \frac{1}{2} a_0 T^2 - \frac{1}{2} a_0 T_S' - \frac{1}{2}\frac{a_2}{a_0} + \frac{3}{8}\frac{a_1^2}{a_0^2}.$$

Now for φ_2/φ_0 in expansion (17) we have

$$\frac{\varphi_2}{\varphi_0} = \frac{157}{900} T^2 + \frac{7}{36} T_S' - \frac{15}{26}(k_1^2 + \delta_1^2) -$$
$$- \frac{7}{36}(k_1 k_2 + \delta_1\delta_2) + \frac{7}{36}(k_{1P}' + \delta_{1Q}') +$$
$$+ \frac{k_1 J_P' + \delta_1 J_Q'}{3J} + \frac{4}{45}\frac{k_2 J_P' + \delta_2 J_Q'}{J} +$$
$$+ \frac{13}{450}\frac{J_P'^2 + J_Q'^2}{J^2} - \frac{4}{45}\frac{J_P'' + J_Q''}{J}. \tag{18}$$

Here the subscripts S, P, Q indicate differentiation with respect to the lengths of the arcs of the curvilinear axes x^1, x^2, x^3,

$$P = \int \sqrt{g_{22}}\, dx^2, \qquad Q = \int \sqrt{g_{33}}\, dx^3,$$

and k_1, k_2 and δ_1, δ_2 are the principal curvatures of the surfaces x^2 = const and x^3 = const, respectively, calculated for $x^1 = 0$

$$k_1 = -\frac{1}{2b_0^{1/2}}\frac{a_{02}'}{a_0}, \qquad k_2 = -\frac{1}{2b_0^{1/2}}\frac{c_{02}'}{c_0},$$
$$\delta_1 = -\frac{1}{2c_0^{1/2}}\frac{a_{03}'}{a_0}, \qquad \delta_2 = -\frac{1}{2c_0^{1/2}}\frac{b_{03}'}{b_0}.$$

Formulas (6)–(9) determine the analytic solution of the equations of a regular electrostatic beam in the presence of emission limited by space charge. Each successive term of the expansion is found from linear algebraic equation (9). These equations, however, quickly become more and more cumbersome. It is advisable, therefore, to turn to high-speed electronic computers to obtain a solution with fairly high accuracy. Thus, we can calculate two-dimensional and three-dimensional flows from a surface of specified form and with a given emission-current density and construct families of equipotential surfaces, each of which can be taken as a collector.

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